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Nonlinear interaction of magnetic field and convection

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The interaction between convection in a horizontal fluid layer heated from below and an ambient vertical magnetic field is considered. The analysis is based on the Boussinesq equations for two-dimensional convection rolls and the assumption that the amplitude A of the convection and the Chandrasekhar number Q are small. It is found that the magnetic energy is amplified by a factor of order $R_m^{\frac{1}{2}}$, where R_m is the magnetic Reynolds number. The ratio between the magnetic and kinetic energies can reach values much larger than unity. Although the magnetic field always inhibits convection, this influence decreases with increasing amplitude of convection. Thus finite amplitude onset of steady convection becomes possible at Rayleigh numbers considerably below the values predicted by linear theory.

1. Introduction

The long-standing interest in the interaction of convection and magnetic field is primarily motivated by astrophysical problems. Observations of the supergranulation pattern of the sun indicate the accumulation of magnetic flux in the boundary regions of the convection cells. The existence of sunspots, on the other hand, indicates the suppression of convection by sufficiently strong magnetic fields. Besides these observable cases of interaction, the theoretical evidence that the solar magnetic field is generated by convection indicates a more intricate connexion between the two phenomena. Since the generation of magnetic flux is accomplished by a complex dynamo process, theoretical analyses usually resort to a statistical description based on assumptions about the small-scale details of the process. Because of the partial expulsion of magnetic fields from turbulent eddies the nature of the interaction between velocity and magnetic fields is far from being well understood. The problem is not restricted to the solar dynamo. Since convection is a likely candidate for the origin of the magnetic fields of the earth and other planets, the problem of interaction of convective eddies with magnetic fields appears in this case in a form similar to that in the solar case.

Although the analysis presented in this paper was originally motivated by the problem of the generation of magnetic fields by convection, we shall restrict our attention to the case of interaction of convection with a homogeneous magnetic field imposed from the outside. Traditionally this problem has been considered from two different points of view. On the one hand, a periodic field of motions similar to those in convection cells has been prescribed and the distortion of the initially homogeneous magnetic field calculated as a function of the magnetic Reynolds number. The papers by Weiss (1966) and Clark (1966) treat this problem in considerable detail. On the other hand, the stabilizing influence of a vertical magnetic field on the onset of convection in a horizontal layer heated from below has been investigated for different physical conditions by linear analysis. An extensive review of this work is given in Chandrasekhar's (1961) monograph.

The difficulties of the nonlinear problem which encompasses both points of view have prevented extensive studies of the general case. The work of Peckover (1971) and Peckover & Weiss (1972) took into account the action of the Lorentz force in a numerical study of forced convection in the presence of a magnetic field. They restricted their attention, however, to the limit of vanishing Prandtl number. Van der Borght, Murphy & Spiegel (1972) have used a mean-field model to describe the influence of a vertical magnetic field on heat transport by convection. More recently Weiss has extended the computations by Peckover and himself to include cases of finite Prandtl number. Some of his as yet unpublished results are mentioned in his review articles (Weiss 1971, 1975). In the present paper a combination of a perturbation method and numerical integration will be used to obtain an explicit dependence of the nonlinear solution on all relevant physical parameters. Although the method of analysis is not adequate to describe some features of the problem in the strongly nonlinear regime, the models appear to be representative for a wide range of parameters.

The analysis starts in § 2 with the formulation of the problem and the solution for the convection velocity field. The solution for the magnetic field is described in § 3. The mathematical analysis of this section is based on a similar analysis carried out in an earlier paper (Busse 1973*a*, referred to as I). In the discussion, § 4, the physical implications of the numerical results are investigated. Some concluding remarks are made in § 5.

2. Mathematical formulation

We consider a horizontal layer of depth d of an electrically conducting fluid heated from below and permeated by a magnetic field. The equations of motion in the Boussinesq approximation, the heat equation and the equation for the magnetic field $\mathbf{B}$ are given by

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\,.\,\nabla\mathbf{v} = -\nabla\frac{p}{\rho_0}-[1-\gamma(T-T_0)]\,\hat{g}\mathbf{k}+\nu\nabla^2\mathbf{v}+\frac{1}{\mu\rho_0}(\nabla\times\mathbf{B})\times\mathbf{B}, \quad (2.1a)$$

$$\nabla\,.\,\mathbf{v} = 0, \quad (2.1b)$$

$$\partial T/\partial t+\mathbf{v}\,.\,\nabla T = \kappa\nabla^2 T, \quad (2.1c)$$

$$\partial \mathbf{B}/\partial t+\mathbf{v}\,.\,\nabla\mathbf{B} = \eta\nabla^2\mathbf{B}+\mathbf{B}\,.\,\nabla\mathbf{v}, \quad (2.1d)$$

where ν, κ and η represent the viscous, thermal and magnetic diffusivity, respectively, and μ is the magnetic permeability. $\hat{g}$ is the acceleration due to gravity and $\mathbf{k}$ is a unit vector in the upward vertical direction. The temperature dependence of the density

$$\rho = \rho_0[1-\gamma(T-T_0)]$$

has been taken into account only in the gravity term according to the Boussinesq approximation. In the static state the temperature T varies linearly between the temperatures $T_0+\frac{1}{2}\Delta T$ and $T_0-\frac{1}{2}\Delta T$ prescribed at the lower and upper boundaries, respectively.

We assume that the magnetic field is uniform in this case and parallel to **k**. We describe the state of convection as a deviation from the static state. Using d as the length scale we introduce non-dimensional Cartesian co-ordinates with the z co-ordinate in the direction of **k** and origin at the bottom of the layer. Restricting our attention to the case of two-dimensional steady convection we define non-dimensional variables ψ, θ and g by writing

$$\mathbf{v} = \frac{\kappa}{d}\left(\frac{\partial \psi}{\partial z}, 0, -\frac{\partial \psi}{\partial x}\right), \quad \mathbf{B} = B_A\left(\frac{\partial g}{\partial z}, 0, -1-\frac{\partial g}{\partial x}\right), \tag{2.2a, b}$$

$$T = T_0 - \Delta T(z-\tfrac{1}{2}) + \Delta T\theta. \tag{2.2c}$$

Note that we have adopted the thermal time scale in the scaling of the velocity field. The non-dimensional equations for the steady problem are readily obtained from (2.1 a, c, d):

$$\nabla^4\psi - R\frac{\partial \theta}{\partial x} = P^{-1}\left(\frac{\partial \psi}{\partial z}\frac{\partial}{\partial x}\nabla^2\psi - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial z}\nabla^2\psi\right) - QS^{-1}\left[\frac{\partial g}{\partial z}\frac{\partial}{\partial x}\nabla^2 g - \left(1+\frac{\partial g}{\partial x}\right)\frac{\partial}{\partial z}\nabla^2 g\right], \tag{2.3a}$$

$$\nabla^2\theta - \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial z}, \tag{2.3b}$$

$$\nabla^2 g = \left(\frac{\partial \psi}{\partial z}\frac{\partial g}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial g}{\partial z} + \frac{\partial \psi}{\partial z}\right)S, \tag{2.3c}$$

where the following non-dimensional parameters have been introduced:

$$R \equiv \frac{\gamma \hat{g}\Delta T d^3}{\nu\kappa}, \quad P \equiv \frac{\nu}{\kappa}, \quad Q \equiv \frac{B_A^2 d^2}{\nu\eta\rho_0\mu}, \quad S \equiv \frac{\kappa}{\eta}. \tag{2.4}$$

R, Q and P are the Rayleigh number, Chandrasekhar number and Prandtl number, respectively. There appears to be no generally accepted name for the parameter S.

We shall consider the case of stress-free boundaries,

$$\partial^2\psi/\partial z^2 = \psi = 0 \quad \text{at} \quad z = 0, 1, \tag{2.5a}$$

and assume that the temperature perturbation θ vanishes at the boundaries:

$$\theta = 0 \quad \text{at} \quad z = 0, 1. \tag{2.5b}$$

In order that the magnetic stress vanishes at the boundaries

$$\partial g/\partial z = 0 \quad \text{at} \quad z = 0, 1 \tag{2.5c}$$

must be assumed. Conditions (2.5) represent the simplest kind of boundary conditions for the problem. They are distinguished by the property that they correspond to a periodic continuation of the convection layer above and below. In the absence of a magnetic field these boundary conditions have long been favoured in the analysis of convection problems. Conditions (2.5) represent the natural extension to the magnetic case. They have been used before by Peckover & Weiss (1972).

We shall solve (2.3) by regarding Q as a perturbation parameter and expanding ψ and R in powers of Q:

$$\psi = \psi_0 + Q\psi_1 + \ldots, \quad R = R_0 + QR_1 + \ldots . \tag{2.6}$$

Analogous expansions hold for θ and g. At zeroth order we obtain the solution without magnetic field, which is well known from earlier work (Malkus & Veronis 1958; Schlüter, Lortz & Busse 1965):

$$\psi_0 = A\{\sin \alpha x \sin \pi z + O(A^2)\}, \tag{2.7a}$$

$$R_0 = (\pi^2+\alpha^2)^3\alpha^{-2} + \tfrac{1}{8}A^2(\pi^2+\alpha^2)^2 + \ldots . \tag{2.7b}$$

We have not included terms of higher order in A since we shall use only the lowest-order representation for ψ_0 in the analysis of the magnetic equation. Comparison with numerical computations of ψ_0 (Fromm 1965) indicates that expressions (2.7) yield a fair approximation even if the second term on the right-hand side of (2.7b) becomes of the same order as the first.

Accordingly, the equation for g_0 is

$$\nabla^2 g_0 - A^*[\pi \cos \pi z \sin \alpha x(1+\partial g_0/\partial x) - \alpha \cos \alpha x \sin \pi z\, \partial g_0/\partial x] = 0, \tag{2.8}$$

where $A^* \equiv AS$. Apart from a factor π, A^* represents the magnetic Reynolds number of the problem based on the maximum horizontal velocity. We shall postpone the solution of (2.8) to the following section and consider the equations at order Q:

$$\nabla^4\psi_1 - R_0\frac{\partial \theta_1}{\partial x} = S^{-1}\left[\frac{\partial g_0}{\partial z}\frac{\partial}{\partial x}\nabla^2 g_0 - \left(1+\frac{\partial g_0}{\partial x}\right)\frac{\partial}{\partial z}\nabla^2 g_0\right] + R_1\frac{\partial \theta_0}{\partial x}, \tag{2.9a}$$

$$\nabla^2\theta_1 - \partial\psi_1/\partial x = 0. \tag{2.9b}$$

In keeping with the approximation discussed above, we have neglected the nonlinear advection terms in these equations. By multiplying (2.9a) by ψ_0 and (2.9b) by $R_0\theta_0$, adding the equations and averaging them over the fluid layer, we obtain the solvability condition

$$O = R_1\left\langle \psi_0\frac{\partial\theta_0}{\partial x}\right\rangle - \left\langle \psi_0\left[\frac{\partial g_0}{\partial z}\frac{\partial}{\partial x}\nabla^2 g_0 - \left(1+\frac{\partial g_0}{\partial x}\right)\frac{\partial}{\partial z}\nabla^2 g_0\right]\right\rangle \tag{2.10}$$

since the left-hand side vanishes after partial integrations have been performed. The second term in this relation represents Ohmic dissipation, which is balanced by the work done by the buoyancy force. Relation (2.10) can be simplified after partial integration and use of (2.8):

$$O = R_1\langle\psi_0\partial\theta_0/\partial x\rangle - S^{-2}\langle|\nabla^2 g_0|^2\rangle. \tag{2.11a}$$

Since g_0 depends only on A^* the evaluation of this equation yields an expression for R_1 of the form

$$R_1 = p(A^*). \tag{2.11b}$$

For the purposes of this paper we shall not be interested in the effects of higher order in Q. Thus the amplitude of convection as a function of the Rayleigh number and the strength of the applied magnetic field is given within our approximation by

$$R - R_\alpha = \tfrac{1}{8}A^2(\pi^2+\alpha^2)^2 + Qp(A^*), \tag{2.12}$$

where $R_\alpha = (\pi^2+\alpha^2)^3\alpha^{-2}$ represents the critical value of the Rayleigh number for the onset of convection rolls with wavenumber α in the absence of a magnetic field. The physical implications of (2.12) will be discussed in § 4 after the function $p(A^*)$ has been determined in § 3.

3. The magnetic field

In order to solve (2.8) we expand g_0 in terms of trigonometric functions which satisfy the boundary condition (2.5*c*):

$$g_0 = \sum_{\nu=1,\,\mu=0}^{\infty} g_{\nu\mu} \sin \nu\alpha x \cos \mu\pi z. \tag{3.1}$$

g_0 is an antisymmetric function of x since the convective motion is symmetric with respect to $x = 0$ and the same symmetry must be assumed for the magnetic field. Multiplication of (2.8) by $\sin n\alpha x \cos m\pi z\,(4-2\delta_{m0})$ yields, after the average over the fluid layer has been taken,

$$A_{nm\nu\mu} g_{\nu\mu} + A^*\pi\delta_{1n}\delta_{1m} = 0, \tag{3.2}$$

where the summation convention is assumed and the definition

$$\begin{aligned} A_{nm\nu\mu} = {} & \tfrac{1}{4}\alpha\pi A^*\{\delta_{\nu,n-1}\delta_{\mu,m-1}(n-m)(1+\delta_{m1}) + \delta_{\nu,n-1}\delta_{\mu,m+1}(n+m) \\ & + \delta_{\nu,n+1}\delta_{\mu,m-1}(-n-m)(1+\delta_{m1}) + \delta_{\nu,n+1}\delta_{\mu,m+1}(-n+m)\} \\ & + (n^2\alpha^2+m^2\pi^2)\,\delta_{\nu n}\delta_{\mu m} \end{aligned} \tag{3.3}$$

has been used. Because of the symmetry of expression (3.3) the equations for the coefficients g_{nm} with even and odd $n+m$ separate. The latter system of equations must have a vanishing solution since an inhomogeneous term is lacking in this case and since generation of magnetic flux cannot take place. We shall solve the system of equations (3.2) numerically for the coefficients g_{nm} with even $n+m$ by assuming that all coefficients with $n+m > N$ are sufficiently small that they can be neglected. This assumption can be tested by replacing N with $N+2$. When the resulting changes in the coefficients g_{nm} are sufficiently small the value N of the truncation parameter is regarded as acceptable. In particular, we shall use the criterion that the energy of the magnetic field changes by less than 1 % if N is replaced with $N+2$.

For the actual calculations we assume $\alpha = \pi$, in which case the problem becomes identical to the corresponding problem solved in I. However, in the present case the calculations have been extended to much larger values of A^* in order to approach the asymptotic range of the solution.

The solution $g_0(x, z)$ is best displayed by plotting magnetic field lines

$$g_0 + x = \text{constant}.$$

Figures 1 (*a*) and (*b*) show typical cases, while a plot for $A^* = 5$ can be found in I. It is remarkable how little the solution changes as A^* is increased from 20 to

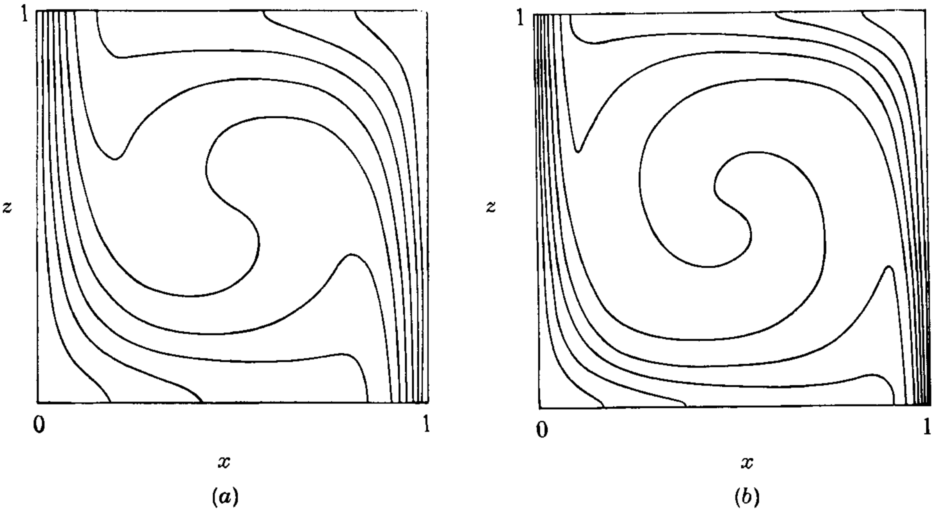

FIGURE 1. Magnetic field lines $x+g = \frac{1}{10}n$ ($n = 1, \ldots, 9$).
(a) $A^* = 20$; (b) $A^* = 60$.

60 except in the boundary layer, where the field strength increases in proportion to $A^{*\frac{1}{2}}$. This suggests the use of boundary-layer theory to obtain an analytical description of the solution. No simple approach for the application of asymptotic methods has been found, however. The fact that the line $g_0+x = \frac{1}{2}$ is rather isolated indicates that the interior of each convection roll is nearly field free. The boundary-layer structure is well displayed when the vertical average of the magnetic field

$$-\frac{\bar{B}_z}{B_A} = 1+\sum_{n=1}^{\frac{1}{2}N} g_{2n0}\,2\pi n\cos 2n\pi x$$

is plotted. Figure 2 shows this function for $A^* = 60$ and 100. It is interesting that the average field actually reverses before vanishing in the interior. This property of the magnetic field is similar to that of the mean vertical temperature gradient in high Rayleigh number convection.

The maximum field strength $B_{\max} = B_A[1+M(A^*)]$ is obviously reached for $x = 0$, $z = 1$ or $x = 1$, $z = 0$. Hence we find

$$M(A^*) = \sum_{\nu=1,\,\mu=0}^{\infty} \pi\nu(-1)^\nu g_{\nu\mu}. \tag{3.4}$$

Figure 3 shows that the function $M(A^*)$ approaches an $A^{*\frac{1}{2}}$ dependence for large values of A^*.

The average magnetic energy density E_M exhibits a similar dependence. Using a non-dimensional formulation for E_M in terms of the Alfvén velocity we write

$$E_M \equiv \frac{B_A^2 d^2}{2\mu\rho\kappa^2}E(A^*).$$

The function $E(A^*)$ is given by

$$E(A^*) = 1+\langle|\nabla g_0|^2\rangle = 1+\frac{1}{4}\sum_{\nu=1,\,\mu=0}^{\infty}(\pi^2\mu^2+\nu^2\alpha^2)\,g_{\nu\mu}^2(1+\delta_{\mu 0}). \tag{3.5}$$

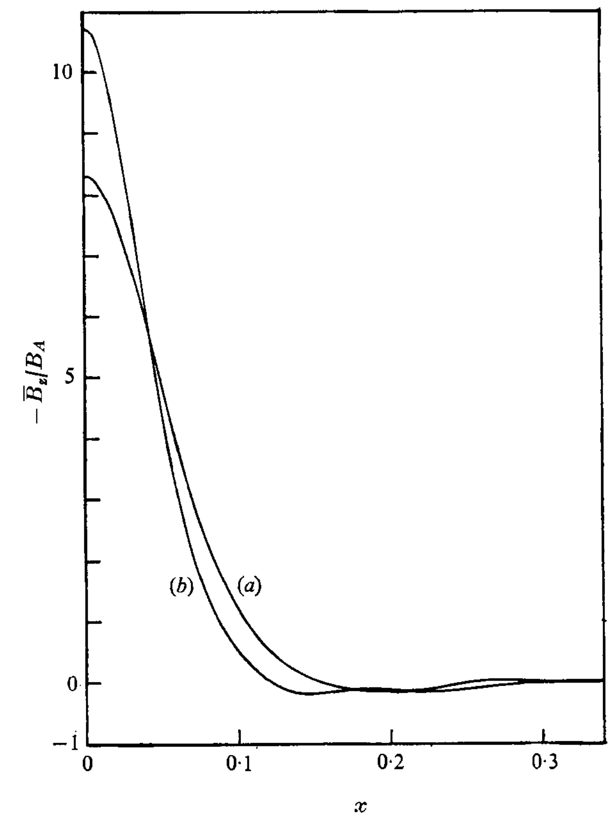

FIGURE 2. The ratio $-\overline{B}_z/B_A$ of the mean vertical field strength to the ambient field strength as a function of the distance from the convection-cell boundary. (*a*) $A^* = 60$. (*b*) $A^* = 100$. The small wiggles in the latter case are expected to disappear in the limit $N \to \infty$.

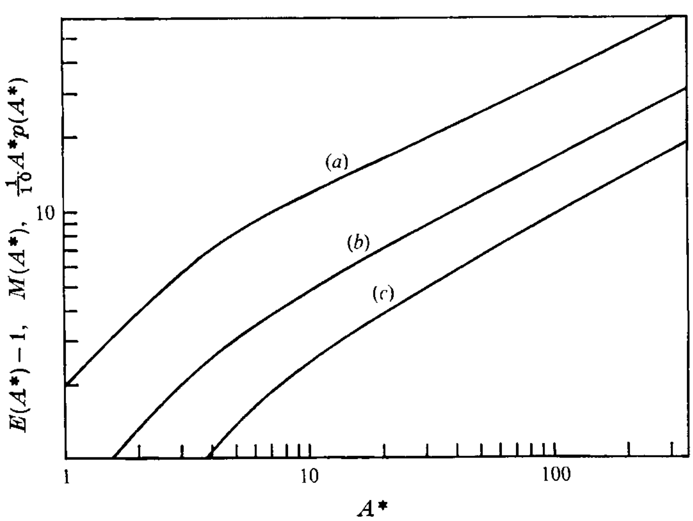

FIGURE 3. (*a*) $\frac{1}{10}A^*p(A^*)$, (*b*) $M(A^*)$ and (*c*) $E(A^*)-1$.

A^*	N	$E(A^*)$	$p(A^*)$	$M(A^*)$
100	22	10·6878	3·45123	16·3493
	24	10·6917	3·45667	16·5234
150	22	13·0180	2·75384	19·3626
	24	13·0553	2·78306	19·7942
	26	13·0724	2·79037	20·1058
200	22	14·8508	2·28342	21·3785
	24	14·9684	2·34665	22·0809
	26	15·0333	2·38300	22·6228
	28	15·0677	2·40296	23·0359
300	26	18·0681	1·82110	25·9170
	28	18·2308	1·87727	26·7293

TABLE 1

It represents the factor by which the magnetic energy is amplified by the action of convection. It is displayed in figure 3 for the case $\alpha = \pi$. Using (2.8), $E(A^*)$ can also be derived in the form

$$E(A^*) = 1 - \tfrac{1}{4}\pi A^* g_{11}.$$

The third function of interest is $p(A^*)$, defined by expressions (2.11). Evaluation of these expressions yields

$$p(A^*) = \frac{\pi^2+\alpha^2}{\alpha^2 A^{*2}} \sum_{\nu=1,\,\mu=0}^{\infty} [(\pi^2\mu^2+\alpha^2\nu^2)\,g_{\nu\mu}]^2\,(1+\delta_{\mu 0}). \tag{3.6}$$

Since $p(A^*)$ decays like $A^{*-\frac{1}{2}}$ for large values of A^*, we have plotted $A^*p(A^*)$ in figure 3.

In table 1 numerical values of the functions $E(A^*)$, $p(A^*)$ and $M(A^*)$ are displayed for various degrees of approximation. The convergence for the two latter functions is not quite as good as that for $E(A^*)$; however, with the exception of the highest value of A^*, it can be regarded as satisfactory.

The dependence of $E(A^*)$, $p(A^*)$ and $M(A^*)$ for high values of A^* suggests an asymptotic power-law dependence of the form

$$E(A^*) = eA^{*\frac{1}{2}}, \quad p(A^*) = rA^{*-\frac{1}{2}}, \quad M(A^*) = mA^{*\frac{1}{2}}. \tag{3.7a}$$

The numerical calculation yields as approximate values for the coefficients

$$e = 1{\cdot}065, \quad r = 34{\cdot}5, \quad m = 1{\cdot}63. \tag{3.7b}$$

As we have mentioned before, the numerical analysis of the present problem is identical in the case $\alpha = \pi$ with the corresponding part of the analysis presented in I. The same computational program was used except for some modifications which were necessary to permit efficient computation of large matrices with ranks up to 210. We wish to use this opportunity to correct expression (5.7) for $p(A^*)$ in I. The correct expression should be identical to expression (3.6) of this paper except that α^2 and π^2 must be exchanged inside the brackets. Although none of the conclusions in I are affected by this error, the form of $p(A^*)$ as plotted in figure 3 of I is slightly changed. The correct dependence of $p(A^*)$ is shown in figure 4 of this paper together with the function $E(A^*)$ for low values of A^*.

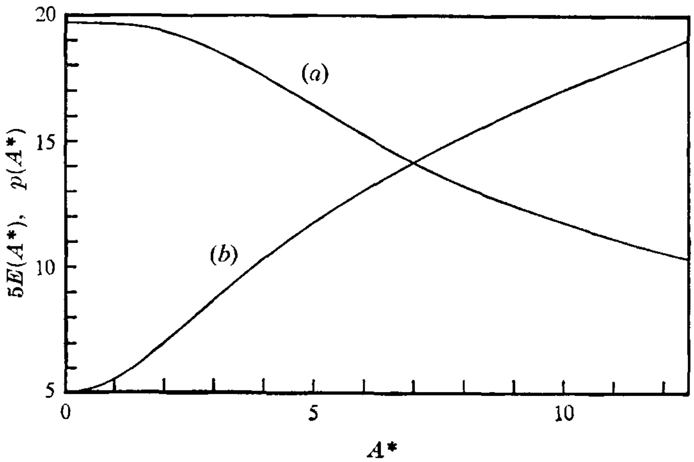

FIGURE 4. (a) $p(A^*)$ and (b) $5E(A^*)$ for low values of A^*.

4. Discussion

The theoretical results derived in this paper are based on the assumption that A and Q are small. These limitations are less severe than one might expect for a number of reasons. For instance, in the limit of vanishing A relation (2.12) reproduces exactly the dependence of the critical Rayleigh number for convection in the presence of a vertical magnetic field which was first derived by Thompson (1951) and Chandrasekhar (1952, 1961). One of the major results of our theory is the fact that the magnetic field changes the amplitude of the velocity field, but not its form to first order, since the term of order Q in (2.12) competes with a term of order A^2 rather than A. This property is borne out by the results of the numerical computations by Peckover & Weiss (1972), which show little change in the form of the velocity field despite large changes in its amplitude. Since, in addition, relation (2.12) provides a good approximation in the non-magnetic case, as we pointed out in § 2, we expect that it will describe the interaction of convection and magnetic field for a much larger range of the relevant parameters than might be anticipated on the basis of the perturbation approach.

The most interesting results of the interaction are obtained when the parameter S becomes large. For values of S much less than unity the ambient homogeneous field is hardly modified by convection and the magnetic field mainly affects the critical Rayleigh number for the onset of convection, as described by the linear theory of Thompson and Chandrasekhar. For increasing values of S the rate $E(A^*)$ at which the magnetic energy is amplified at a given supercritical value of the Rayleigh number increases rapidly. For the special value $R = 2R_\pi$ we have plotted the amplification rate in figure 5 as a function of Q. Asymptotically $E(A^*)$ increases in proportion to $S^{\frac{1}{2}}$ for sufficiently low values of Q.

The increase $M(A^*)$ in the maximum field strength above the ambient value behaves similarly to the function $E(A^*)-1$. This is because the area in which most of the field is concentrated decreases in the same manner as $M(A^*)^{-1}$. Hence

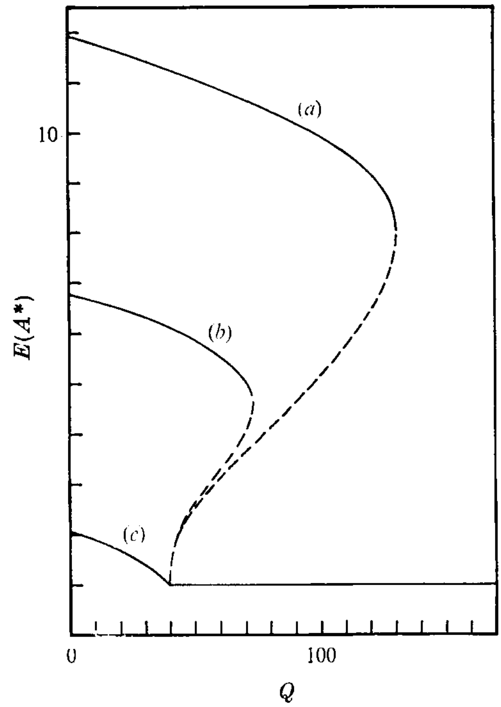

FIGURE 5. $E(A^*)$ as a function of the Chandrasekhar number Q in the special case $R = 2R_\pi = 16\pi^4$. – – –, steady solution is unstable. (a) $S = 30$; (b) $S = 10$; (c) $S = 1$.

there is little need to discuss the dependence of the maximum field strength separately. Of more interest is the dependence of the magnetic energy density E_M on Q, which is given according to (3.5) by

$$E_M = \tfrac{1}{2}QE(A^*)/SP. \tag{4.1}$$

E_M first increases linearly with the ambient energy density until it reaches a maximum value $E_{M\,\max}$ if S is sufficiently large. From its maximum value E_M decreases with increasing Q until it drops to the ambient value, when the convective motion stops. To calculate $E_{M\,\max}$ we use the asymptotic expressions (3.7) for $E(A^*)$ and $p(A^*)$. We first determine the value A^* at which $QA^{*\frac{1}{2}}$ reaches its maximum. From

$$A^*dQ + \tfrac{1}{2}QdA^* = 0$$

and
$$(\pi^4 A^* S^{-2} - \tfrac{1}{2}rA^{*-\frac{3}{2}}Q)\,dA^* + p(A^*)\,dQ = 0,$$

which follows from (2.12) with $\alpha = \pi$, we find

$$A_m^* = [(2S^2/3\pi^4)(R-R_\pi)]^{\frac{1}{2}}. \tag{4.2}$$

The corresponding value of $E_M(A^*)$ is

$$E_{M\,\max} = (e/2r\pi^2)\,P^{-1}[\tfrac{2}{3}(R-R_\pi)]^{\frac{3}{2}}. \tag{4.3}$$

It is of interest to compare this result with the average kinetic energy density E_K of the convection, which for the amplitude (4.2) assumes the value

$$E_{Km} = (R - R_\pi)/3\pi^2. \tag{4.4}$$

The ratio

$$E_{M\,\mathrm{max}}/E_{Km} = (e/r)\,P^{-1}[\tfrac{2}{3}(R - R_\pi)]^{\frac{1}{2}} \tag{4.5}$$

between magnetic and kinetic energy obviously can attain large values even within the limitations of our theory. The limit of vanishing Prandtl number should be regarded with caution, however. The two-dimensional form of convection is not physically realizable in this limit because of the oscillatory instability (Busse 1972). The form of this instability indicates that it will not be stabilized by the presence of the magnetic field.

So far we have emphasized the action of the convective motion on the magnetic field at a given value of the Rayleigh number. It is just as interesting to take the opposite point of view and consider the effect of a given imposed magnetic field on the state of convection. It is obvious from expression (2.12) that the magnetic field always exerts a stabilizing influence on convection in that it increases the Rayleigh number in comparison with the non-magnetic case. However, the stabilizing influence diminishes as the amplitude of convection increases, since $p(A^*)$ is a monotonically decreasing function of A^*. This changes the nature of the dependence of A on the Rayleigh number; in place of the monotonic dependence in the non-magnetic case, a reversal of the dependence becomes possible. The Rayleigh number at finite amplitudes drops below the critical value determined by linear stability analysis and subcritical instability becomes possible.

To calculate the lowest value R_{min} of the Rayleigh number at which convection can exist we again use the asymptotic expressions (3.7) for $p(A^*)$ in relation (2.12). Setting $\alpha = \pi$ we find as a necessary condition for R_{min}

$$A^*\pi^4 S^{-2} - \tfrac{1}{2} r A^{*-\frac{3}{2}} Q = 0. \tag{4.6}$$

This relation for A^* yields

$$R_{\mathrm{min}} = R_\pi + \tfrac{5}{2}(\tfrac{1}{2} r\pi Q/\sqrt{S})^{\frac{4}{5}}. \tag{4.7}$$

This result gives a power law for the dependence of Q different from the linear result in the case $A = 0$,

$$R_c = R_\pi + 2\pi^2 Q, \tag{4.8}$$

to which we referred above. Accordingly, steady convection can occur at significantly lower Rayleigh numbers than those predicted by the linear theory. In figure 6 we have plotted the value R_{min} for the onset of subcritical finite amplitude convection for different values of S. This figure shows that the asymptotic form (4.7) is valid for most of the region of interest. The diminishing influence of the magnetic field at finite amplitudes is caused by flux expulsion from the interior of the convection roll. This phenomenon is, in a sense, opposite to the alignment process suggested by Malkus (1959), in which the Lorentz force counteracts the stabilizing effect of the nonlinear momentum advection terms. The latter process is likely to be relevant when the convection layer rotates about a vertical axis.

When the time dependence in the equations of convection is taken into account it becomes obvious that the steady solution described by expression (2.12) is

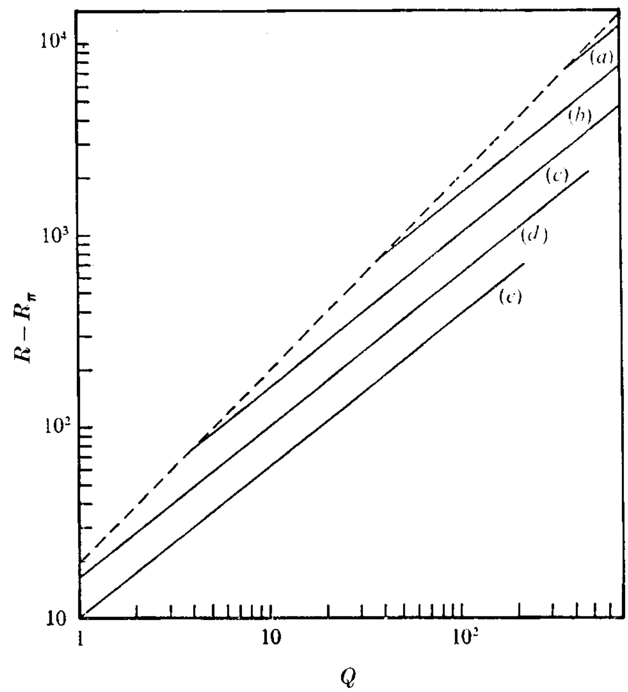

FIGURE 6. The lowest Rayleigh number R_{min} at which steady convection is possible as a function of the Chandrasekhar number Q. – – –, result of the linear theory. (a) $S = 1$; (b) $S = 3$; (c) $S = 10$; (d) $S = 30$; (e) $S = 100$.

unstable for values of A^* less than that given by (4.6). This is to be expected, since the Rayleigh number decreases with increasing amplitude in this region. Because of this instability the dashed portion of the dependence of $E(A^*)$ shown in figure 4 cannot be realized physically.

In the case $S \geqslant 1$ oscillatory as well as steady convection can occur in the presence of a sufficiently strong magnetic field, as discussed by Chandrasekhar (1961). Although the critical Rayleigh number for oscillatory convection is lower than that for steady convection, it also exhibits a linear dependence on Q in contrast to expression (4.7). Both oscillatory convection and finite amplitude subcritical steady convection require a finite value of Q. In the latter case the threshold value Q_t which must be exceeded in order that R reaches a minimum value at a finite value of A^* is given by

$$Q_t = \pi^4 S^{-2} \min\{A^*/-p'(A^*)\} \approx 3{\cdot}23\pi^4 S^{-2},$$

corresponding to $A^* \approx 3{\cdot}2$. Since both Q and A require finite values for subcritical steady convection for values of S of order unity, it cannot be established rigorously within the framework of the present theory that finite amplitude steady convection can occur at Rayleigh numbers below the value for oscillatory onset. It is well known from other convection problems, for instance, the problem of thermohaline convection (Veronis 1968), that finite amplitude effects of oscillatory convection are rather small. Thus we expect that expression (4.7) governs the onset of steady convection even in the case when oscillatory convection sets in at a lower Rayleigh number. At values Q of order 10^4 and higher this conclusion may have to be modified since computations by Weiss (1975) indicate a persistence of the oscillatory mode at high convection amplitudes. Obviously, the parameter regime of these calculations lies beyond the validity of our approximation.

In a discussion of the onset of convection the dependence of the Rayleigh number on the wavenumber should be taken into account. We have neglected this important point since computations with varying wavenumber become prohibitively expensive. We expect, however, that the value $\alpha = \pi$ which we have chosen in the present work is representative in all qualitative aspects and that the results will change only quantitatively for other values of the wavenumber.

5. Concluding remarks

It is a popular hypothesis to assume equipartition between magnetic and kinetic energy for processes in electrically conducting fluids. An assumption of this kind is justified in the case of Alfvén waves and related magnetohydrodynamic processes for which dissipation is of minor importance. In cases such as convection, which are dominated by dissipative effects, the equipartition hypothesis is not valid, as is demonstrated by the results of this paper and the earlier results of Peckover & Weiss (1972). Magnetic and kinetic energies play a secondary role in convection as compared with the rates of Ohmic, viscous and thermal dissipation. Accordingly the diffusivity ratios become the important parameters of the physical processes. In the particular case treated in this paper magnetic and viscous dissipation compete. Because of the magnetic-flux expulsion process Ohmic dissipation leads to a more complicated relationship between amplitude and Rayleigh number than in the case of pure viscous and thermal dissipation.

The fact that magnetic energy is amplified by the convective motion must not be interpreted as a dynamo process. As is evident from figure 1, magnetic flux is a conserved quantity in the problem considered in this paper. A dynamo process which generates magnetic flux requires dependence on the third spatial co-ordinate. In I it is shown how an additional component of the velocity field in the third direction can lead to magnetic-field generation.

The property that in the present case as well as in I the velocity field does not depend on the co-ordinate in the third direction has the advantage that the analysis remains unchanged in a system rotating about an axis in that direction. The Coriolis force, which represents a dominating influence in rapidly rotating systems such as the earth's core, enforces nearly two-dimensional convection. Hence the problem treated in this paper is likely to be relevant to the interaction of convection and the magnetic field in the earth's core.

The fact that the inhibition of finite amplitude convection by a magnetic field is much less than that predicted by linear theory may have some bearing on the physics of sunspots. Although the basic magnetostatic balance in sunspots is independent of the field strength in simple models (Busse 1973*b*), the fact that weak magnetic fields do not affect convection much for values of S as high as those in the solar photosphere appears to be responsible for the absence of sunspots with a magnetic field strength below the order of 10^3 gauss.

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REFERENCES

Busse, F. H. 1972 *J. Fluid Mech.* **52**, 97.
Busse, F. H. 1973*a* *J. Fluid Mech.* **57**, 529.
Busse, F. H. 1973*b* *Solar Phys.* **33**, 413.
Chandrasekhar, S. 1952 *Phil. Mag.* **43** (7), 501.
Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*, chap. 4. Oxford: Clarendon Press.
Clark, A. 1966 *Phys. Fluids*, **9**, 485.
Fromm, J. E. 1965 *Phys. Fluids*, **8**, 1757.
Malkus, W. V. R. 1959 *Astrophys. J.* **130**, 259.
Malkus, W. V. R. & Veronis, G. 1958 *J. Fluid Mech.* **4**, 225.
Peckover, R. S. 1971 Ph.D. thesis, University of Cambridge.
Peckover, R. S. & Weiss, N. O. 1972 *Comp. Phys. Comm.* **4**, 339.
Schlüter, A., Lortz, D. & Busse, F. 1965 *J. Fluid Mech.* **28**, 223.
Thompson, W. B. 1951 *Phil. Mag.* **42** (7), 1417.
Van der Borght, R., Murphy, J. D. & Spiegel, E. A. 1972 *Austr. J. Phys.* **25**, 703.
Veronis, G. 1968 *J. Fluid Mech.* **34**, 315.
Weiss, N. O. 1966 *Proc. Roy. Soc.* A **293**, 310.
Weiss, N. O. 1971 In *Solar Magnetic Fields* (ed. R. F. Howard), p. 757. Dordrecht: Reidel.
Weiss, N. O. 1975 *Adv. in Chem. Phys.* **32** (in Press).